BAA008 Matematika I (G)

Cvičení č. 6

Příklad 7.3.4. Užitím smíšeného součinu rozhodněte, zda vektory $\vec{u}, \vec{v}, \vec{w}$ jsou komplanární, jestliže $\vec{u} = \vec{a} - 2\vec{b} + \vec{c}, \vec{v} = 3\vec{a} + \vec{b} - 2\vec{c}, \vec{w} = 7\vec{a} + 14\vec{b} - 13\vec{c}$, kde $\vec{a}, \vec{b}, \vec{c}$ jsou nekomplanární vektory.

$$[\vec{u}, \vec{v}, \vec{w}] = [\vec{v}, \vec{w}, \vec{u}] = [\vec{w}, \vec{u}, \vec{v}]$$

$$[\vec{u}, \vec{v}, \vec{w}] = \vec{u}\,(\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{a} - 2\vec{b} + \vec{c}) \cdot [(3\vec{a} + \vec{b} - 2\vec{c}) \times$$

$$\times (7\vec{a} + 14\vec{b} - 13\vec{c})] = (\vec{a} - 2\vec{b} + \vec{c}) \cdot (21 \underbrace{\cancel{\vec{a} \times \vec{a}}}_{= \vec{o}} + 42\, \vec{a} \times \vec{b} -$$

$$- 39\, \vec{a} \times \vec{c} + 7 \underbrace{\vec{b} \times \vec{a}}_{-7\, \vec{a} \times \vec{b}} + 14 \underbrace{\cancel{\vec{b} \times \vec{b}}}_{= \vec{o}} - 13\, \vec{b} \times \vec{c} - 14 \underbrace{\vec{c} \times \vec{a}}_{+14\, \vec{a} \times \vec{c}} - 28 \underbrace{\vec{c} \times \vec{b}}_{+28\, \vec{b} \times \vec{c}} +$$

$$+ 26 \underbrace{\cancel{\vec{c} \times \vec{c}}}_{= \vec{o}} = (\vec{a} - 2\vec{b} + \vec{c}) \cdot (35\, \vec{a} \times \vec{b} - 25\, \vec{a} \times \vec{c} + 15\, \vec{b} \times \vec{c}) =$$

$$= 35\, \cancel{\vec{a} \cdot (\vec{a} \times \vec{b})} - 25\, \cancel{\vec{a} \cdot (\vec{a} \times \vec{c})} + 15\, \vec{a} \cdot (\vec{b} \times \vec{c}) - 70\, \cancel{\vec{b} \cdot (\vec{a} \times \vec{b})} +$$

$$+ 50\, \vec{b} \cdot (\vec{a} \times \vec{c}) - 30\, \cancel{\vec{b} \cdot (\vec{b} \times \vec{c})} + 35\, \vec{c} \cdot (\vec{a} \times \vec{b}) - 25\, \cancel{\vec{c} \cdot (\vec{a} \times \vec{c})} +$$

$$+ 15\, \cancel{\vec{c} \cdot (\vec{b} \times \vec{c})} = 15\,[\vec{a}, \vec{b}, \vec{c}] + 50\,[\vec{b}, \vec{a}, \vec{c}] + 35\,[\vec{c}, \vec{a}, \vec{b}] =$$

$$= 15\,[\vec{a}, \vec{b}, \vec{c}] - 50\,[\vec{a}, \vec{b}, \vec{c}] - 35\,[\vec{a}, \vec{b}, \vec{c}] = 0$$

$\Rightarrow$ $\vec{u}, \vec{v}, \vec{w}$ JSOU KOMPLANÁRNÍ

Příklad 7.4.2. Užitím vztahu $(I) : \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \cdot \vec{c}) - \vec{c} \cdot (\vec{a} \cdot \vec{b})$ dokažte, že platí

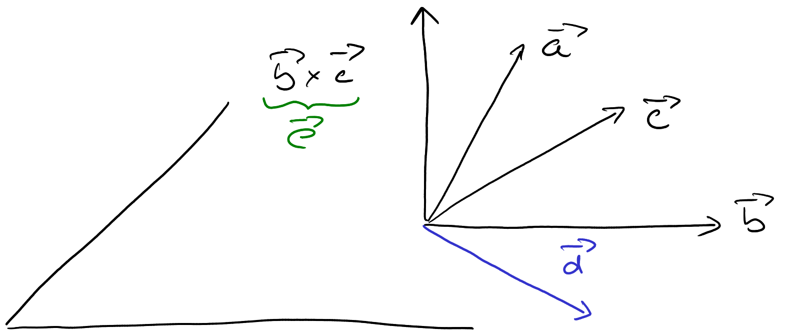
a) $(\vec{b} \times \vec{c}) \cdot [(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})] = [\vec{a}, \vec{b}, \vec{c}]^2$,

b) $\vec{a} \times [\vec{b} \times (\vec{c} \times \vec{d})] = [\vec{a} \cdot (\vec{c} \times \vec{d})] \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot (\vec{c} \times \vec{d})$,

c) $\vec{a} \times [\vec{b} \times (\vec{c} \times \vec{d})] = (\vec{b} \cdot \vec{d}) \cdot (\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c}) \cdot (\vec{a} \times \vec{d})$,

d) $(\vec{a} \times \vec{b}) \cdot [(\vec{c} \times \vec{d}) \times (\vec{e} \times \vec{f})] = [(\vec{a} \times \vec{b}) \cdot \vec{e}] \cdot [\vec{f} \cdot (\vec{c} \times \vec{d})] - [(\vec{a} \times \vec{b}) \cdot \vec{f}] \cdot [\vec{e} \cdot (\vec{c} \times \vec{d})]$.

$$(I)\quad \vec a\times(\vec b\times\vec c) = \underset{\text{VEKTOR}}{\underset{\downarrow}{\vec b}}\cdot\underbrace{(\vec a\cdot\vec c)}_{\in\mathbb{R}} - \underset{\text{VEKTOR}}{\underset{\downarrow}{\vec c}}\underbrace{(\vec a\cdot\vec b)}_{\in\mathbb{R}}$$



$$\underbrace{\vec a\times\underbrace{(\vec b\times\vec c)}_{\vec e}}_{\vec d}$$

$\vec d$ LEŽÍ V ROVINĚ $(\vec b,\vec c)$

DŮLEŽITÉ IDENTITY

$\vec a\times(\vec b\times\vec c) = (\vec a\cdot\vec c)\vec b - (\vec a\cdot\vec b)\vec c$

$(\vec a\times\vec b)\cdot(\vec c\times\vec d) = (\vec a\cdot\vec c)\cdot(\vec b\cdot\vec d) - (\vec b\cdot\vec c)(\vec a\cdot\vec d)$

$(\vec a\times\vec b)\times(\vec c\times\vec d) = [\vec a,\vec b,\vec d]\cdot\vec c - [\vec a,\vec b,\vec c]\cdot\vec d$

a) $(\vec b\times\vec c)\cdot[\underbrace{(\vec c\times\vec a)\times(\vec a\times\vec b)}] = [\vec a,\vec b,\vec c]^2$

$$(\underbrace{\vec c\times\vec a})\times(\vec a\times\vec b) \overset{(I)}{=} \vec a((\vec c\times\vec a)\cdot\vec b) - \vec b((\vec c\times\vec a)\cdot\vec a) =$$
$$= \vec a\cdot[\vec c,\vec a,\vec b] - \vec b[\underbrace{\vec c,\vec a,\vec a}_{=0}] =$$
$$= \vec a\cdot[\vec a,\vec b,\vec c]$$

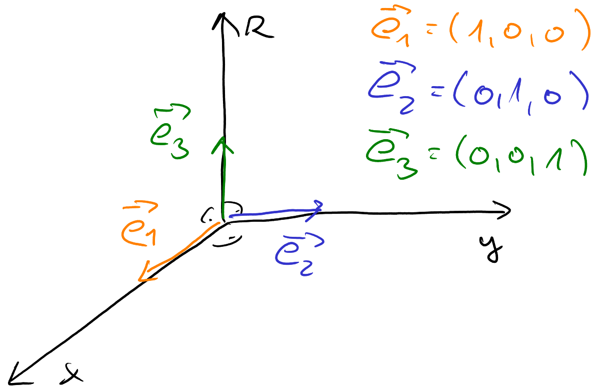
$$\underbrace{(\vec b\times\vec c)\cdot\vec a}\,[\vec a,\vec b,\vec c] = [\vec b,\vec c,\vec a]\cdot[\vec a,\vec b,\vec c] = \underline{\underline{[\vec a,\vec b,\vec c]^2}}$$

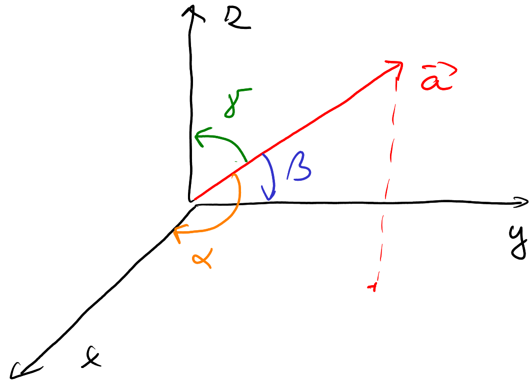
b), c), d) DÚ

Příklad 7.5.1. Určete směrové kosiny vektorů

a) $\vec{a} = \vec{e}_2 + 3\vec{e}_3$,

b) $\vec{b} = 2\vec{e}_1 - 3\vec{e}_2 - \vec{e}_3$.





$(\cos\alpha, \cos\beta, \cos\gamma) = \vec{a}_0$ ... JEDNOTKOVÝ VEKTOR

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 = \frac{a_1^2 + a_2^2 + a_3^2}{\|\vec{a}\|^2}$$

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = (a_1, a_2, a_3)$$

$$\vec{a}\cdot\vec{e}_1 = \|\vec{a}\|\cdot\|\vec{e}_1\|\cdot\cos\alpha \Rightarrow \cos\alpha = \frac{\vec{a}\cdot\vec{e}_1}{\|\vec{a}\|\cdot\|\vec{e}_1\|} = \frac{a_1}{\|\vec{a}\|}$$

PODOBNĚ: $\cos\beta = \dfrac{a_2}{\|\vec{a}\|}$ $\quad \cos\gamma = \dfrac{a_3}{\|\vec{a}\|}$

a) $\vec{a} = \vec{e}_2 + 3\vec{e}_3 = (0, 1, 3)$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{e}_1}{\|\vec{a}\|\cdot\|\vec{e}\|} = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}} = \frac{0}{\sqrt{0+1+9}} = \frac{0}{\sqrt{10}} = \underline{\underline{0}}$$

$$\cos\beta = \frac{a_2}{\|\vec{a}\|} = \underline{\underline{\frac{1}{\sqrt{10}}}} \qquad \cos\gamma = \frac{a_3}{\|\vec{a}\|} = \underline{\underline{\frac{3}{\sqrt{10}}}} \qquad \vec{a}_0 = \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

b) $\vec{b} = 2\vec{e}_1 - 3\vec{e}_2 - \vec{e}_3 = (2, -3, -1)$ $\quad \|\vec{b}\| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{14}$

$$\cos\alpha = \underline{\underline{\frac{2}{\sqrt{14}}}}, \quad \cos\beta = \underline{\underline{-\frac{3}{\sqrt{14}}}}, \quad \cos\gamma = \underline{\underline{-\frac{1}{\sqrt{14}}}} \qquad \vec{b}_0 = \left(\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\right)$$

Příklad 7.5.2. Najděte jednotkový vektor $\vec{c}^0$ kolmý k vektorům $\vec{a} = 2\vec{e}_1 - \vec{e}_2 + \vec{e}_3, \vec{b} = \vec{e}_1 + 2\vec{e}_2 - \vec{e}_3$.

I.) „LEPŠÍ"

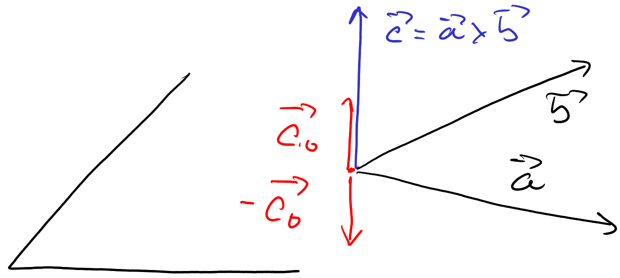
$\vec{a} = (2,-1,1)$, $\vec{b} = (1,2,-1)$

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} \leftarrow LR \; = \vec{e}_1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{e}_2 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{e}_3 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} =$$

$$= (-1, 3, 5)$$

$$\|\vec{c}\| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}$$

$$\vec{c}_0 = \frac{\vec{c}}{\|\vec{c}\|} = \pm \frac{1}{\sqrt{35}}(-1, 3, 5)$$



$$\vec{c}_{01} = \left(-\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right), \quad \vec{c}_{02} = \left(\frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}, -\frac{5}{\sqrt{35}}\right)$$

II) $\left.\begin{array}{l} \vec{a} \cdot \vec{c} = 0 \\ \vec{b} \cdot \vec{c} = 0 \end{array}\right\}$ SYSTÉM 2 ROVNIC O TŘECH NEZNÁMÝCH $\Rightarrow \infty$ ŘEŠENÍ $\Rightarrow$ 1 VOLNÝ PARAMETR

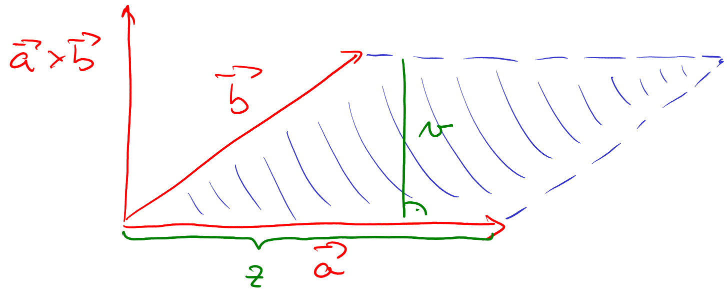
$$\vec{c} = (c_1, c_2, c_3)$$

$$\begin{array}{l} 2c_1 - c_2 + c_3 = 0 \\ c_1 + 2c_2 - c_3 = 0 \end{array} \Bigg| \; c_3 = 1 \; \left.\right\} \begin{array}{ll} 2c_1 - c_2 = -1 & /\cdot 2 \\ c_1 + 2c_2 = 1 & \end{array} \quad 5c_1 = -1 \Rightarrow c_1 = -\frac{1}{5}$$

$$c_2 = \frac{3}{5}$$

$$\vec{c} = \left(-\frac{1}{5}, \frac{3}{5}, 1\right) \rightsquigarrow (-1, 3, 5) \rightsquigarrow (1, -3, -5) \Rightarrow \vec{c}_0 = \pm \frac{1}{\sqrt{35}}(1, -3, -5)$$

Příklad 7.5.3. Vypočtěte plošný obsah a výšku na stranu určenou vektorem $\vec{a}$ rovnoběžníka setrojeného nad vektory $\vec{a} = 2\vec{e}_2 + \vec{e}_3, \vec{b} = \vec{e}_1 + 2\vec{e}_3$.



$$P = \|\vec{a} \times \vec{b}\|$$

$$P = z \cdot v$$

1. $\vec{a} = (0,2,1)$, $\vec{b} = (1,0,2)$

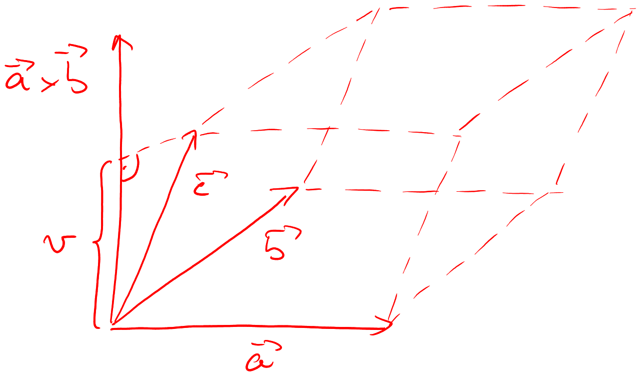
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = (4, 1, -2)$$

$$P = \|\vec{a} \times \vec{b}\| = \|(0,2,1) \times (1,0,2)\| = \|(4,1,-2)\| = \sqrt{4^2 + 1^2 + (-2)^2} = \underline{\underline{\sqrt{21}}}$$

2. $P = z \cdot v$ $\qquad z = \|\vec{a}\| = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}$

$$v = \frac{P}{z} = \frac{\sqrt{15}}{\sqrt{5}} = \underline{\underline{\sqrt{\frac{15}{5}}}}$$

Příklad 7.5.4. Vypočtěte objem rovnoběžnostěnu sestrojeného nad vektory $\vec{a}, \vec{b}, \vec{c}$, plošný obsah stěny sestrojené nad vektory $\vec{a}, \vec{b}$ a velikost výšky na tuto stěnu, je-li $\vec{a} = 3\vec{e}_1 + 2\vec{e}_2$, $\vec{b} = 2\vec{e}_1 + 3\vec{e}_2$, $\vec{c} = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$.



$$v = \|\vec{c}_{\vec a \times \vec b}\| = \|\vec c\| \cdot \cos(\vec c, \vec a \times \vec b) = \|\vec c\| \cdot \frac{\vec c \cdot (\vec a \times \vec b)}{\|\vec c\| \cdot \|\vec a \times \vec b\|} = \frac{\vec c \cdot (\vec a \times \vec b)}{\|\vec a \times \vec b\|}$$

$$V = |[\vec a, \vec b, \vec c]|, \quad P = \|\vec a \times \vec b\|, \quad V = P \cdot v \Rightarrow v = \frac{V}{P}$$

1. $\vec a = (3,2,0)$, $\vec b = (2,3,0)$, $\vec c = (1,2,3)$

$$V = |[\vec a, \vec b, \vec c]| = \left| \begin{vmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{vmatrix} \right| = 3 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = \underline{\underline{15}}$$

$\uparrow$ LR

2. $$\vec a \times \vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 3 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = \vec i \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} - \vec j \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} + \vec k \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = (0,0,5)$$

$$P = \|\vec a \times \vec b\| = \|0,0,5\| = \sqrt{0^2 + 0^2 + 5^2} = \sqrt{25} = \underline{\underline{5}}$$

3. $$v = \frac{V}{P} = \frac{15}{5} = \underline{\underline{3}}$$

Příklad 7.5.5. Určete úhel vektorů $\vec{a} = \vec{e}_1 + \vec{e}_2 - 4\vec{e}_3, \vec{b} = \vec{e}_1 - 2\vec{e}_2 + 2\vec{e}_3$.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\varphi \implies \cos\varphi = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}$$

$$\vec{a} = (1, 1, -4)$$
$$\vec{b} = (1, -2, 2)$$

$$\vec{a} \cdot \vec{b} = (1,1,-4) \cdot (1,-2,2) = 1 \cdot 1 + 1 \cdot (-2) + (-4) \cdot 1 = -9$$
$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + 1^2 + (-4)^2} = \sqrt{18} = 3\sqrt{2}$$
$$\|\vec{b}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} = \frac{-9}{3 \cdot \sqrt{2} \cdot 3} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\implies \varphi = \frac{3}{4}\pi = 135^\circ$$

Příklad 7.5.6. Vypočtěte $\vec{a}\times(\vec{b}\times\vec{c})$, kde $\vec{a}=2\vec{e}_1, \vec{b}=3\vec{e}_2, \vec{c}=\vec{e}_1+\vec{e}_3$

a) přímým výpočtem,

b) užitím vztahu $(I) : \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{c}\cdot(\vec{a}\cdot\vec{b})$,

c) užitím determinantů.

POZNÁMKA:

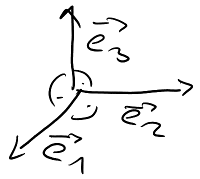
$A\cdot B\cdot C$ (matice) ✓ SPRÁVNÝ ZÁPIS

$\vec{a}\cdot\vec{b}\cdot\vec{c}$ ✗ NESPRÁVNÝ Z. } NUTNO

$\vec{a}\times\vec{b}\times\vec{c}$ ✗ NESPRÁVNÝ Z } ZÁVORKOVAT!

a) $$\vec{a}\times(\vec{b}\times\vec{c}) = 2\vec{e}_1\times(3\vec{e}_2\times(\vec{e}_1+\vec{e}_3)) = 2\vec{e}_1(3\underbrace{\vec{e}_2\times\vec{e}_1}_{-\vec{e}_3}+3\underbrace{\vec{e}_2\times\vec{e}_3}_{\vec{e}_1}) =$$

$$= -6\underbrace{\vec{e}_1\times\vec{e}_3}_{-\vec{e}_2} + 6\underbrace{\vec{e}_1\times\vec{e}_1}_{\vec{o}} = 6\vec{e}_2$$



b) (I) $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) = 3\vec{e}_2(2\vec{e}_1\cdot(\vec{e}_1+\vec{e}_3)) -$$

$$- \underbrace{(\vec{e}_1+\vec{e}_3)\cdot(\underbrace{2\vec{e}_1\cdot 3\vec{e}_2}_{=0})}_{=\vec{o}} = 3\vec{e}_2(\underbrace{2\vec{e}_1\cdot\vec{e}_1}_{2\cdot\|\vec{e}\| = 2\cdot 1} + \underbrace{2\vec{e}_1\cdot\vec{e}_3}_{=0}) =$$

$$= 6\vec{e}_2$$

c) $$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \vec{e}_1\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} - \vec{e}_2\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + \vec{e}_3\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = 3\vec{e}_1 - 3\vec{e}_3$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & 0 & 0 \\ 3 & 0 & -3 \end{vmatrix} = \vec{e}_1\begin{vmatrix} 0 & 0 \\ 0 & -3 \end{vmatrix} - \vec{e}_2\begin{vmatrix} 2 & 0 \\ 3 & -3 \end{vmatrix} + \vec{e}_3\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 6\vec{e}_2$$