BAA008 Matematika I (G)

Cvičení č. 12

2. zápočtová písemka

Příklad 12.2.1. Vypočítejte s použitím L'Hospitalova pravidla:

$$\lim_{x\to 0} \frac{\sin 3x^2}{\ln \cos(2x^2 - x)}.$$

L'HOSPITALOVO PRAVIDLO

$x_0 \in \mathbb{R} \cup \{-\infty, \infty\}$, $f(x), g(x)$ - DIFERENCOVATELNÉ

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0 \ \vee \ \lim_{x\to x_0} |g(x)| = \infty$$

$$\boxed{\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}}$$

$$\lim_{x\to 0} \frac{\sin 3x^2}{\ln \cos(2x^2 - x)} \left|\frac{0}{0}\right| \overset{LP}{=} \lim_{x\to 0} \frac{\cos 3x^2 \cdot 6x}{\frac{-\sin(2x^2-x)\cdot(4x-1)}{\cos(2x^2-x)}} =$$

$$= \lim_{x\to 0} \frac{6x \cdot \cos 3x^2 \cdot \cos(2x^2-x)}{(1-4x)\cdot \sin(2x^2-x)} \left|\frac{0}{0}\right| \overset{*}{=}$$

$$\left.\begin{aligned} &\lim_{x\to x_0} f(x) = r, \quad \lim_{x\to x_0} g(x) = s \\ &\lim_{x\to x_0} \big(f(x)\cdot g(x)\big) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = r \cdot s \end{aligned}\right\} *$$

$$= \lim_{x\to 0} \frac{6 \cdot \cos 3x^2 \cdot \cos(2x^2-x)}{1-4x} \cdot \lim_{x\to x_0} \frac{x}{\sin(2x^2-x)} \overset{LP}{=}$$

$$= 6 \cdot \lim_{x\to x_0} \frac{1}{\cos(2x^2-x)\cdot(4x-1)} = \underline{\underline{-6}}$$

Příklad 12.2.x. Vypočítejte s použitím L'Hospitalova pravidla:

$$\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right).$$

$$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -1\cdot x^{-2}$$

$$\left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)' = -2x^{-3}$$

$$\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right)\left|\infty\cdot 0\right| = \lim_{x\to\infty}\frac{1-\cos\frac{1}{x}}{\frac{1}{x^2}}\left|\frac{0}{0}\right| \overset{LP}{=}$$

$$= \lim_{x\to\infty}\frac{\sin\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}} = \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{2}{x}}\left|\frac{0}{0}\right| = \lim_{x\to\infty}\frac{\cos\frac{1}{x}\left(\cancel{-}\frac{1}{\cancel{x^2}}\right)}{\cancel{-}\frac{2}{\cancel{x}}} =$$

$$= \lim_{x\to\infty}\frac{\cos\frac{1}{x}}{2} = \underline{\underline{\frac{1}{2}}}$$

Příklad 13.2. Určete Taylorův polynom $T_n(F, A, \mathrm{d}x)$, $n \in \mathbb{N}$, $A = [-2]$, je-li $f : y = \ln(3 - 4x)$.

$$T_n(f, x_0, h) = f(x_0) + \frac{d f(x_0, h)}{(1!)} + \frac{d^2 f(x_0, h)}{2!} + \dots + \frac{d^n f(x_0, h)}{n!}$$

$$h = (x - x_0)$$

$$d^n f(x_0, h) = f^{n}(x_0) \cdot h^n = f^n(x_0) \cdot (x - x_0)^n$$

$$y' = \frac{-4}{3-4x} = -4(3-4x)^{-1}$$

$$y'' = -4 \cdot (-1)(3-4x)^{-2}(-4) = -4 \cdot 4(3-4x)^{-2} = -4^2(3-4x)^{-2} = -\frac{1 \cdot 4^2}{(3-4x)^2}$$

$$y''' = -4^2(-2)(3-4x)^{-3} \cdot (-4) = -2 \cdot 4^3 (3-4x)^{-3} = -\frac{1 \cdot 2 \cdot 4^3}{(3-4x)^3}$$

$$y^{IV} = -2 \cdot 4^3 (-3)(3-4x)^{-4}(-4) = -2 \cdot 3 \cdot 4^4 (3-4x)^{-4} = -\frac{2 \cdot 3 \cdot 4^4}{(3-4x)^4}$$

$$y^{V} = -2 \cdot 3 \cdot 4^4 (-4)(3-4x)^{-5}(-4) = -2 \cdot 3 \cdot 4 \cdot 4^5 (3-4x)^{-5} = -\frac{2 \cdot 3 \cdot 4 \cdot 4^5}{(3-4x)^5}$$

$$y^{VI} = -2 \cdot 3 \cdot 4 \cdot 4^5 (-5)(3-4x)^{-6}(-4) = -2 \cdot 3 \cdot 4 \cdot 5 \cdot 4^6 (3-4x)^{-6} = -\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 4^6}{(3-4x)^6}$$

$\downarrow$ ⋮

$$y^n = -\frac{(n-1)!\, 4^n}{(3-4x)^n}$$

$$f(-2) = \ln 11, \quad f'(-2) = -\frac{4^2}{11^2}, \quad f''(-2) = -2\frac{4^3}{11^3}, \quad \dots, \quad f^n(-2) = -(n-1)!\,\frac{4^n}{11^n}$$

$$df(-2, (x+2)) = f'(-2)(x+2) = -\frac{4}{11}(x+2)$$

$$d^2 f(-2, (x+2)) = f''(-2)(x+2)^2 = -\frac{4^2}{11^2}(x+2)^2$$

$$d^3 f(-2, (x+2)) = f'''(-2)(x+2)^3 = -\frac{1 \cdot 2 \cdot 4^3}{11^3}(x+2)^3$$

⋮

$$d^n f(-2, (x+2)) = f^n(-2)(x+2)^n = -\frac{(n-1)! \cdot 4^n}{11^n}(x+2)^n$$

$$T_n = \ln 11 - \frac{4}{11}(x+2) - \frac{4^2}{2 \cdot 11^2}(x+2)^2 - \underbrace{\frac{\cancel{2}\, 4^3}{\cancel{2}3 \cdot 11^3}(x+2)^3}_{\frac{4^3}{3 \cdot 11^3}(x+2)^3} - \dots - \underbrace{\frac{\cancel{(n-1)!}\, 4^n}{n\cancel{!}\, 11^n}(x+2)^n}_{\frac{4^n}{n\, 11^n}(x+2)^n} =$$

$$= \ln 11 - \sum_{k=1}^{n} \frac{4^k}{k \cdot 11^k}(x+2)^k$$

$$\boxed{T_3 = \ln 11 - \frac{4}{11}(x+2) - \frac{4^2}{2 \cdot 11^2}(x+2)^2 - \frac{4^3}{3 \cdot 11^3}(x+2)^3}$$

Příklad 13.4.x. Určete Maclaurinův polynom $n-$tého stupně v bodě $x_0 = 0$ funkce

$$f : y = x \cdot e^x.$$

MACLAURINŮV POLYNOM

$x_0 = 0 \qquad (x - x_0) = x \qquad d^n f(0,h) = f^n(0) \cdot x^n$

$y = x \cdot e^x$ $\qquad y(0) = 0$

$y' = e^x + xe^x$ $\qquad y'(0) = 1$

$y'' = e^x + e^x + xe^x = 2e^x + xe^x$ $\qquad y''(0) = 2$

$y''' = 2e^x + e^x + xe^x = 3e^x + xe^x$ $\qquad y'''(0) = 3$

$y^{IV} = 3e^x + e^x + xe^x = 4e^x + xe^x$ $\qquad y^{IV}(0) = 4$

$\vdots$

$y^n = ne^x + xe^x$ $\qquad y^n(0) = n$

$$T_n(0) = M_n(0) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

$$\cdots + \frac{f^n(0)}{n!} x^n = 0 + \frac{1}{1!} x + \underbrace{\frac{2}{2!}}_{\frac{1}{1!}} x^2 + \underbrace{\frac{3}{3!}}_{\frac{1}{2!}} x^3 + \underbrace{\cdots}_{\frac{1}{3!}\cdots} + \underbrace{\frac{n}{n!}}_{\frac{1}{(n-1)!}} x^n$$

$$= \sum_{k=1}^{n} \frac{1}{(k-1)!} x^k$$