BAA009 Matematika 2 (G)

Cvičení č. 7

Příklad 7.1. Určete parciální derivace 3. řádu funkce $z = x^2 \sin^2 y$.

$$[\ z'''_{xxx} = 0,\ z'''_{xxy} = z'''_{xyx} = z'''_{yxx} = 2\sin 2y,$$
$$z'''_{xyy} = z'''_{yxy} = z'''_{yyx} = 4x\cos 2x,\ z'''_{yyy} = -4x^2 \sin 2y\ ]$$

$z = x^2 \cdot \sin^2 y$

$z'_x = 2x \sin^2 y$

$z'_y = x^2 \cdot 2 \sin y \cos y = x^2 \sin 2y$

$z''_{xx} = 2 \sin^2 y$

$z''_{xy} = 2x\, 2 \sin y \cos y = 2x \sin 2y$

$z''_{yx} = 2x \sin 2y$

$z''_{yy} = x^2 \cos 2y \cdot 2 = 2x^2 \cos 2y$

$z'''_{xxx} = 0$

$z'''_{xxy} = z'''_{xyx} = z'''_{yxx} = 2 \sin 2y$

$z'''_{xyy} = 2 \cos 2y\, 2x = 4x \cos 2y$

$z'''_{yyy} = 2x^2 (-\sin 2y)\, 2 = -4x^2 \sin 2y$

Příklad 7.2. Určete parciální derivace prvního řádu funkce $f(x, y, z) = x^3+8x^2y+6y^3z^4+9y-4z+7$ v bodě $A = [1, 2, -1]$.

[$f'_x = 35$, $f'_y = 89$, $f'_z = -196$]

$f(x,y,z) = x^3 + 8x^2y + 6y^3z^4 + 9y - 4z + 7$, $A[1,2,-1]$

$f'_x = 3x^2 + 8y\,2x + 0 + 0 - 0 + 0 = 3x^2 + 16xy$

$f'_y = 0 + 8x^2 \cdot 1 + 6z^4\,3y^2 + 9 \cdot 1 - 0 + 0 = 8x^2 + 18y^2z^4 + 9$

$f'_z = 0 + 0 + 6y^3\,4z^3 + 0 - 4 \cdot 1 + 0 = 24y^3z^3 - 4$

$f'_x(A) = 35$

$f'_y(A) = 89$

$f'_z(A) = -196$

Příklad 7.3. Určete parciální derivace 2. řádu funkce $f(x, y, z) = (x^2 + 3y)e^{4yz}$.

[$f''_{xx} = 2e^{4yz}$, $z''_{xy} = z''_{yx} = 8xze^{4yz}$, $f''_{xz} = f''_{zx} = 8xye^{4yz}$, $f''_{yy} = (16x^2z^2 + 48yz^2 + 24z)e^{4yz}$, $f''_{yz} = f''_{zy} = (4x^2 + 24y + 16x^2yz + 48y^2z)e^{4yz}$, $f''_{zz} = (16x^2y^2 + 48y^3)e^{4yz}$]

$$f'_x = 2x \cdot e^{4yz}$$

$$f'_y = 3 \cdot e^{4yz} + (x^2+3y)e^{4yz} \cdot 4z = (3 + 4x^2z + 12yz)e^{4yz}$$

$$f'_z = (x^2+3y)e^{4yz} \cdot 4y = (4x^2y + 12y^2)e^{4yz}$$

$$f''_{xx} = 2e^{4yz}$$

$$f''_{xy} = f''_{yx} = 2x e^{4yz} \cdot 4z = 8xze^{4yz}$$

$$f''_{xz} = f''_{zx} = 2x e^{4yz} \cdot 4y = 8xye^{4yz}$$

$$f''_{yx} = (0 + 4z\,2x + 0)e^{4yz} = 8xze^{4yz}$$

$$f''_{yy} = 12z \cdot e^{4yz} + (3 + 4x^2z + 12yz)e^{4yz} \cdot 4z = (24z + 16x^2z^2 + 48yz^2)e^{4yz}$$

$$f''_{yz} = f''_{zy} = (4x^2 + 24y)e^{4yz} + (4x^2y + 12y^2)e^{4yz} \cdot 4z =$$

$$= (4x^2 + 24y + 16x^2yz + 48y^2z)e^{4yz}$$

$$f''_{zz} = (4x^2y + 12y^2)e^{4yz} \cdot 4y = (16x^2y^2 + 48y^3)e^{4yz}$$

Příklad 7.4. Určete $\mathrm{d}f$, $\mathrm{d}^2 f$ funkce $f(x,y) = e^x \cos y$ v bodě $A = [0,0]$.

[$\mathrm{d}f = x$, $\mathrm{d}^2 f = x^2 - y^2$]

DIFERENCIÁL V BODĚ $A = [x_0, y_0]$

1. ŘÁDU : $df(A) = f'_x(A)\underbrace{\overbrace{(x-x_0)}^{h}}_{dx} + f'_y(A)\underbrace{\overbrace{(y-y_0)}^{k}}_{dy}$

2. ŘÁDU : $d^2 f(A) = f''_{xx}(A)(x-x_0)^2 + 2 f''_{xy}(A)(x-x_0)(y-y_0) + f''_{yy}(A)(y-y_0)^2$

3. ŘÁDU : $d^3 f(A) = f'''_{xxx}(A)(x-x_0)^3 + 3 f'''_{xxy}(A)(x-x_0)^2(y-y_0) +$
$+ 3 f'''_{xyy}(A)(x-x_0)(y-y_0)^2 + f'''_{yyy}(A)(y-y_0)^3$

$f(x,y) = e^x \cos y$, $A[0,0]$

$f'_x = e^x \cos y$ $\qquad f'_x(A) = e^0 \cdot \cos 0 = 1 \cdot 1 = 1$

$f'_y = e^x(-\sin y) = -e^x \sin y$ $\qquad f'_y(A) = -e^0 \sin 0 = -1 \cdot 0 = 0$

$df(A) = \underset{1}{f'_x(A)}(x-0) + \underset{0}{f'_y(A)}(y-0) = 1 \cdot x + 0 \cdot y = \underline{\underline{x}}$

$f''_{xx} = e^x \cos y$ $\qquad f''_{xx}(0,0) = 1 \cdot 1 = 1$

$f''_{xy} = f''_{yx} = -e^x \sin y$ $\qquad f''_{xy}(0,0) = -1 \cdot 0 = 0$

$f''_{yy} = -e^x \cos y$ $\qquad f''_{yy}(0,0) = -1 \cdot 1 = -1$

$d^2 f(A) = 1(x-0)^2 + 0(x-0)(y-0) - 1(y-0)^2 = \underline{\underline{x^2 - y^2}}$

1	$(a+b)^1$
1 1	$(a+b)^2$
1 2 1	$(a+b)^3$
1 3 3 1	$\vdots$
1 4 6 4 1	

Příklad 7.5. Určete Taylotův polynom stupně n funkce f v bodě A

a) $f(x,y) = y \ln x$, $A = [1,2]$, $n = 2$,

[$T_2(A) = 2(x-1) - (x-1)^2 + (x-1)(y-2)$]

TAYLORŮV POLYNOM STUPNĚ n FUNKCE $z = f(x,y)$ V BODĚ $A[x_0, y_0]$

$$T_n(A) = f(A) + \frac{df(A)}{1!} + \frac{d^2f(A)}{2!} + \dots + \frac{d^nf(A)}{n!}$$

$$f(X) \doteq T_n(A) + \cancel{R_n(A)}$$

$A = [0,0] \rightarrow$ MACLAURINŮV POLYNOM

$z = y \ln x$, $A[1,2]$, $n = 2$

$f'_x = y \cdot \frac{1}{x} = \frac{y}{x} = y x^{-1}$

$f'_y = \ln x$

$f''_{xx} = y \cdot (-1) x^{-2} = -\frac{y}{x^2}$

$f''_{xy} = \frac{1}{x}$

$f''_{yy} = 0$

$f(A) = 0$

$f'_x(A) = 2$

$f'_y(A) = 0$

$f''_{xx}(A) = -2$

$f''_{xy}(A) = 1$

$f''_{yy}(A) = 0$

$$df(A) = f'_x(A)(x-x_0) + f'_y(A)(y-y_0) = 2(x-1) + 0(y-2) = 2(x-1)$$

$$d^2f(A) = f''_{xx}(A)(x-x_0)^2 + 2f''_{xy}(A)(x-x_0)(y-y_0) + f''_{yy}(A)(y-y_0)^2 =$$
$$= -2(x-1)^2 + 2 \cdot 1 (x-1)(y-2) + 0(y-2)^2 = -2(x-1)^2 + 2(x-1)(y-2)$$

$$T_2(A) = 0 + 2(x-1) + \frac{-2(x-1)^2 + 2(x-1)(y-2)}{2!} =$$

$$= \underline{\underline{2(x-1) - (x-1)^2 + (x-1)(y-2)}}$$

b) $f(x,y) = 3x^2y + \sin^2 x + 5y - 2$, $A = [0,0]$, $n = 3$.

[$M_3(A) = -2 + 5y + x^2 + 3x^2y$]

$$T_3(A) = f(A) + \frac{df(A)}{1!} + \frac{d^2f(A)}{2!} + \frac{d^3f(A)}{3!}$$

$f(A) = -2$

$f'_x = 6xy + 2\sin x\cos x = 6xy + \sin 2x$

$f'_y = 3x^2 + 5$

$f'_x(A) = 0$

$f'_y(A) = 5$

$df(A) = 0(x-0) + 5(y-0) = 5y$

$f''_{xx} = 6y + \cos 2x \cdot 2 = 6y + 2\cos 2x$

$f''_{xy} = 6x$

$f''_{yy} = 0$

$f''_{xx}(A) = 2$

$f''_{xy}(A) = 0$

$f''_{yy}(A) = 0$

$d^2f(A) = 2(x-0)^2 + 2\cdot 0(x-0)(y-0) + 0(y-0)^2 = 2x^2$

$f'''_{xxx} = 2(-\sin 2x)2 = -4\sin 2x$

$f'''_{xxy} = 6$

$f'''_{xyy} = 0$

$f'''_{yyy} = 0$

$f'''_{xxx}(A) = 0$

$f'''_{xxy}(A) = 6$

$f'''_{xyy}(A) = 0$

$f'''_{yyy}(A) = 0$

$d^3(A) = f'''_{xxx}(A)(x-x_0)^3 + 3f'''_{xxy}(A)(x-x_0)^2(y-y_0) + 3f'''_{xyy}(A)(x-x_0)(y-y_0)^2 + f'''_{yyy}(A)(y-y_0)^3$

$= \cancel{0(x-0)^3} + 3\cdot 6(x-0)^2(y-0) + \cancel{3\cdot 0(x-0)(y-0)^3} + \cancel{0(y-0)^3}$

$= 18x^2y$

$$T_3(A) = -2 + 5y + \frac{2x^2}{2!} + \frac{18x^2y}{3!} = \underline{\underline{-2 + 5y + x^2 + 3x^2y}}$$