BAA009 Matematika 2 (G)

Cvičení č. 10

Příklad 10.1. Určete tečnou rovinu a normálu v bodě $T = [2, -1, ?]$ plochy $f : z = x^2 + 2y^2$.

[$\tau : 4x - 4y - z - 6 = 0; n : x = 2 + 4t, y = -1 - 4t, z = 6 - t, t \in \mathbb{R}$]

TEČNÁ ROVINA A NORMÁLA

$z = f(x,y)$: τ : $z - z_0 = f'_x(T)(x - x_0) + f'_y(T)(y - y_0)$

$f'_x(T)(x - x_0) + f'_y(T)(y - y_0) - (z - z_0) = 0$

n : $x = x_0 + f'_x(T)\, t$

$y = y_0 + f'_y(T)\, t$ $\quad t \in \mathbb{R}$

$z = z_0 - t$

$\vec{s}_n = \vec{n}_\tau = (f'_x(T), f'_y(T), -1)$

$z = x^2 + 2y^2$, $T = [2, -1, ?]$ $\qquad z_T = 4 + 2 = 6 \rightarrow T = [2, -1, 6]$

$z'_x = 2x \qquad z'_x(T) = 4$

$z'_y = 4y \qquad z'_y(T) = -4$

τ : $f'_x(x - x_0) + f'_y(y - y_0) - (z - z_0) = 0$

$4(x - 2) - 4(y + 1) - (z - 6) = 0$

$4x - 4y - z - 6 = 0 \qquad \rightarrow \vec{n}_\tau = \vec{s}_n = (4, -4, -1)$

n : $x = 2 + 4t$

$y = -1 - 4t \qquad t \in \mathbb{R}$

$z = 6 - t$

Příklad NP Určete tečnou rovinu a normálu v bodě $A = [2, -6, ?]$ plochy $z = f(x, y)$ zadané implicitní rovnicí $F : x^2 + y^2 + z^2 - 49 = 0$.

$[\ \tau_1 : 2x - 6y - 3z - 49 = 0;\ n_1 : x = 2 + 2t, y = -6 - 6t, z = -3 - 3t, t \in \mathbb{R};$

$[\ \tau_2 : 2x - 6y + 3z - 49 = 0;\ n_2 : x = 2 + 2s, y = -6 - 6s, z = 3 + 3s, s \in \mathbb{R}\ ]$

$\tau: \quad F_x'(T)\,(x-x_0) + F_y'(T)\,(y-y_0) + F_z'\,(z-z_0) = 0$

$n: \quad x = x_0 + F_x'(T)\, t$

$\quad\ \ y = y_0 + F_y'(T)\cdot t \qquad t \in \mathbb{R}$

$\quad\ \ z = z_0 + F_z'(T)\cdot t$

$F(x,y,z): \quad x^2 + y^2 + z^2 - 49 = 0 \quad , A = [2, -6, ?]$

$A \in F(x,y,z): \quad 4 + 36 + z^2 - 49 = 0$

$z^2 = 9 \quad \Rightarrow z_{1,2} = \pm 3$

$A_1 = [2, -6, -3]$

$A_2 = [2, -6, 3]$

$F_x' = 2x \qquad F_x'(A_1) = 4 \qquad F_x'(A_2) = 4$

$F_y' = 2y \qquad F_y'(A_1) = -12 \qquad F_y'(A_2) = -12$

$F_z' = 2z \qquad F_z'(A_1) = -6 \qquad F_z'(A_2) = 6$

$\Rightarrow \vec{n}_1 = (4, -12, -6) \rightsquigarrow (2, -6, -3)$

$\vec{n}_2 = (4, -12, 6) \rightsquigarrow (2, -6, 3)$

$A_1: \quad \tau_1: \quad 2(x-2) - 6(y+6) - 3(z+3) = 0$

$2x - 6y - 3z - 49 = 0$

$n_1: \quad x = 2 + 2t$

$\quad\ \ y = -6 - 6t \qquad t \in \mathbb{R}$

$\quad\ \ z = -3 - 3t$

A_2: DÚ ...

Příklad 10.2. Určete tečnou rovinu plochy $f : z = \operatorname{arctg}\frac{y}{x}$, která je rovnoběžná s rovinou ρ : $x - y + 2z - 1 = 0$.

$$[\,\tau : x - y + 2z - \frac{\pi}{2} = 0\,]$$

NORMÁLOVÝ VEKTOR ϱ: $\vec{n_\varrho} = (1,-1,2)$, PRO $\vec{n_\varrho}$ PLATÍ, ŽE $\vec{n_\varrho} = \alpha \cdot \vec{n_\tau}$, PROTOŽE $\tau \parallel \varrho$.

$$z_x' = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = -\frac{y}{x^2+y^2}$$

$$z_y' = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2}$$

BOD DOTYKU OZNAČME $T=[x_0, y_0, z_0]$

POTOM PLATÍ:

$$z_x' = \frac{-y_0}{x_0^2+y_0^2} \quad ; \quad z_y' = \frac{x_0}{x_0^2+y_0^2}$$

PRO NORMÁLOVÝ VEKTOR $\vec{n_\tau}$ PLATÍ: $\vec{n_\tau} = (z_x', z_y', -1)$,

TJ.

$$1 = \alpha\, z_x'(T) \qquad -1 = \alpha \cdot z_y'(T) \qquad 2 = \alpha(-1)$$

$$1 = \alpha \frac{-y_0}{x_0^2+y_0^2} \qquad -1 = \alpha \cdot \frac{x_0}{x_0^2+y_0^2} \qquad -2 = \alpha$$

$$1 = \frac{2y_0}{x_0^2+y_0^2} \qquad -1 = \frac{-2x_0}{x_0^2+y_0^2}$$

$$\Rightarrow \begin{array}{l} x_0^2 + y_0^2 = 2y_0 \\ -x_0^2 - y_0^2 = -2y_0 \end{array} \Big) + \quad \Rightarrow x_0 = y_0$$

... DOSADÍME DO 1. RCE:

$$y_0^2 + y_0^2 - 2y_0 = 0$$
$$2y_0(y_0 - 1) = 0$$

$\Rightarrow$ 1. ŘEŠENÍ: $y_0 = 0$, $x_0 = 0$ ... V x_0 NENÍ z DEFINOVANÁ, TEDY NEMÁ ŘEŠENÍ

2. ŘEŠENÍ: $y_0 = 1$, $x_0 = 1$, $z_0 = \frac{\pi}{4}$, $z_x'(T) = -\frac{1}{2}$, $z_y'(T) = \frac{1}{2}$

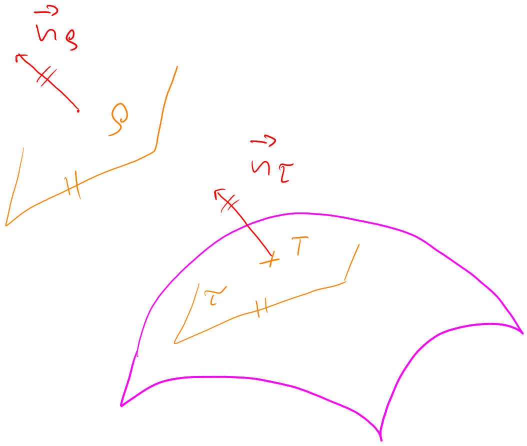
$$\tau: \; z - \frac{\pi}{4} = -\frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

$$0 = x - y + 2z - \frac{\pi}{2}$$

Příklad NP Určete tečnou rovinu a normálu plochy $z = f(x,y)$ určené implicitní rovnicí $F: 3x^2 + 2y^2 + z^2 - 21 = 0$, která je rovnoběžná s rovinou $\rho : 6x + 4y + z = 0$.

[$\tau_1 : 6x + 4y + z - 21 = 0$; $n_1 : x = 2 + 6t, y = 2 + 4t, z = 1 + t, t \in \mathbb{R}$;

[$\tau_2 : 6x + 4y + z + 21 = 0$; $n_2 : x = -2 + 6s, y = -2 + 4s, z = -1 + s, s \in \mathbb{R}$]



$\vec{n}_\tau = (F_x'(T), F_y'(T), F_z'(T))$

$\vec{n}_\tau = t \cdot \vec{n}_\rho$

$F_x' = 6x$ $\quad \vec{n}_\rho = (6,4,1)$

$F_y' = 4y$ $\quad \vec{n}_\tau = (6x, 4y, 2z) = (6t, 4t, t)$

$F_z' = 2z$

$$\left.\begin{array}{l} 6x = 6t \\ 4y = 4t \\ 2z = t \end{array}\right\} \quad \begin{array}{l} x = t \\ y = t \\ z = \frac{1}{2}t \end{array}$$

$\Rightarrow T[t, t, \frac{1}{2}t]$

$F(T): \quad 3t^2 + 2t^2 + \frac{1}{4}t^2 - 21 = 0 \quad / \cdot 4$

$12t^2 + 8t^2 + t^2 - 21 \cdot 4 = 0$

$\cancel{21}t^2 = \cancel{21} \cdot 4 \quad \Rightarrow t = \pm 2 \quad \Rightarrow$ $T_1 = [2, 2, 1]$, $T_2 = [-2, -2, -1]$

T_1: $\quad \tau_1: 6x + 4y + z + d = 0$

$T \in \tau: \quad 12 + 8 + 1 + d = 0$

$d = -21$

$\tau_1: \quad 6x + 4y + z - 21 = 0$

$n_1: \quad x = 2 + 6s$

$y = 2 + 4s \qquad s \in \mathbb{R}$

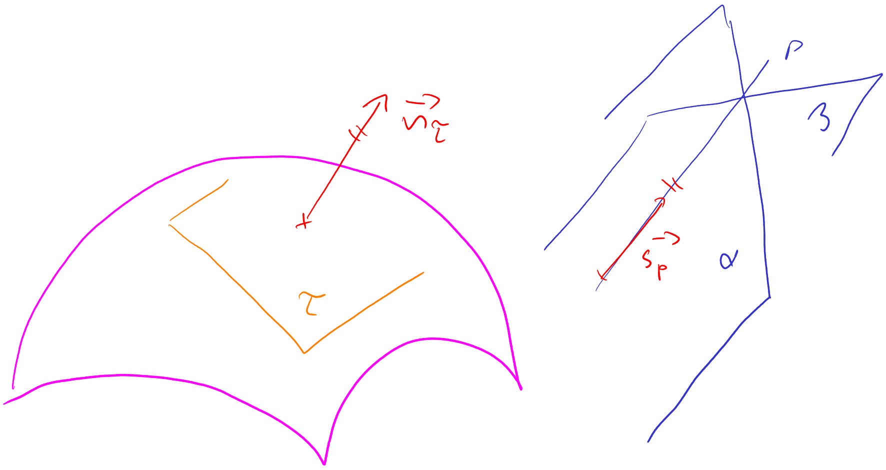
$z = 1 + s$

T_2: DÚ ...

Příklad NP Určete tečnou rovinu a normálu plochy $z = f(x,y)$ určené implicitní rovnicí $F: (x-1)^2 + y^2 + z^2 - 2 = 0$, která je kolmá k rovinám $\alpha: 2x - 2y - z - 3 = 0$ a $\beta: x - y - z = 0$.

[$\tau_1: x + y - 3 = 0$; $n_1: x = 2 + t, y = 1 + t, z = 0, t \in \mathbb{R}$;

[$\tau_2: x + y + 1 = 0$; $n_2: x = s, y = -1 + s, z = 0, s \in \mathbb{R}$]



$F'_x = 2(x-1)$ $\qquad \vec{n}_\alpha = (2,-2,-1)$ $\qquad \vec{s}_p = \vec{n}_\alpha \times \vec{n}_\beta$

$F'_y = 2y$ $\qquad \vec{n}_\beta = (1,-1,-1)$ $\qquad \vec{n}_\tau = t \cdot \vec{s}_p$

$F_z = 2z$

$$\vec{s}_p = \vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ 1 & -1 & -1 \end{vmatrix} = (2-1)\vec{i} - (-2+1)\vec{j} + (-2+2)\vec{k} = (1,1,0)$$

$\rightarrow (2(x-1), 2y, 2z) = (t, t, 0)$

$$\left.\begin{aligned} 2(x-1) &= t \quad \Rightarrow \quad x = 1 - \tfrac{1}{2}t \\ 2y &= t \quad \Rightarrow \quad y = \tfrac{1}{2}t \\ 2z &= 0 \quad \Rightarrow \quad z = 0 \end{aligned}\right\} T = \left[1 + \tfrac{t}{2}, \tfrac{t}{2}, 0\right]$$

$F(T)$: $\left(1 + \frac{t}{2} - 1\right)^2 + \left(\frac{t}{2}\right)^2 - 2 = 0$

$\frac{t^2}{4} + \frac{t^2}{4} - 2 = 0$

$t^2 = 4 \Rightarrow t = \pm 2$

$T_1 = [2, 1, 0]$

$T_2 = [0, -1, 0]$

T_1: τ_1: $x + y + d = 0$ $\qquad n$: $x = 2 + s$

$T_1 \in \tau_1$: $2 + 1 + d = 0$ $\qquad y = 1 + s \qquad s \in \mathbb{R}$

$d = -3$ $\qquad z = 0$

τ_1: $x + y - 3 = 0$

T_2: DÚ ....

Příklad 10.3. Najděte tečnu a obecnou rovnici normálové roviny prostorové křivky

$$\gamma : \begin{cases} x = t^4 + t^2 + 1 \\ y = 4t^3 + 5t + 2 \\ y = t^4 - t \end{cases} \quad \text{v bodě } T = [3, -7, 2].$$

[$t : x = 3 - 6s, y = -7 + 17s, z = 2 - 5s, s \in \mathbb{R}; \eta : -6x + 17y - 5z + 147 = 0$]

k: 1. $x = t^4 + t^2 + 1$ $\quad T = [3, -7, 2]$

2. $y = 4t^4 + 5t + 2$

3. $z = t^4 - t$

1.: $t^4 + t^2 + 1 = 3 \Rightarrow t^4 + t^2 - 2 = 0 \Rightarrow (t^2+2)(t^2-1) = 0 \Rightarrow$

$\Rightarrow t_{1,2} = \pm 1$

3: $t^4 - t = 2$, $\quad t_1 = -1 : 1 + 1 = 2$ ✓ ; $\quad t_2 = 1 : 1 - 1 = 0$ × $\quad \Rightarrow t_0 = -1$

[LZE DOSADIT I DO 2., OPĚT VYJDE $t_0 = -1$]

$x'(t) = 4t^3 + 2t$ $\qquad x'(t_0) = -6$

$y'(t) = 12t^2 + 5$ $\qquad y'(t_0) = 17$

$z'(t) = 4t^3 - 1$ $\qquad z'(t_0) = -5$

t: $x = 3 - 6s$

$y = -7 + 17s$, $\quad s \in \mathbb{R}$

$z = 2 - 5s$

μ: $-6(x-3) + 17(y+7) - 5(z-2) = 0$

$-6x + 18 + 17y + 119 - 5z + 10 = 0$

$-6x + 17y - 5z + 147 = 0$

Příklad NP Určete tečnu a normálovou rovinu prostorové křivky γ, která je dána jako průečnice ploch $\gamma: \begin{cases} F: -x^2 - y^2 + z = 0 \\ G: -x + y = 0 \end{cases}$ v bodě $P(x_0 = 2)$.

[$t: x = 2 - s, y = 2 - s, z = 8 - 8s, s \in \mathbb{R}; \nu : x + y + 8z - 68 = 0$]

$\Rightarrow$ z $G(P) \rightarrow y_0 = 2$

$\Rightarrow$ z $F(P) \rightarrow x_0 = 8$ $\Rightarrow P[2, 2, 8]$

$F: \; -x^2 - y^2 - z = 0$

$G: \; -x + y = 0$

$F'_x = -2x \qquad F'_x(P) = -4 \qquad G'_x = -1 \qquad G'_x(P) = -1$

$F'_y = -2y \qquad F'_y(P) = -4 \qquad G'_y = 1 \qquad G'_y(P) = 1$

$F_z = 1 \qquad F'_z(P) = 1 \qquad G'_z = 0 \qquad G'_z(P) = 0$

SMĚROVÝ VEKTOR TEČNY
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & -4 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (0-1)\vec{i} - (0-1)\vec{j} + (-4-4)\vec{k} = (-1, -1, -8)$$

$t: \quad x = 2 - t$
$\quad\; y = 2 - t \qquad t \in \mathbb{R}$
$\quad\; z = 8 - 8t$

$\nu: 1(x-2) + 1(y-2) + 8(z-8) = 0$

$x + y + 8z - 68 = 0$

Příklad 10.4. Určete tečny ke křivce $\gamma : x = \frac{1}{4}t^4, y = \frac{1}{3}t^3, z = \frac{1}{2}t^2$ a rovnoběžné s rovinou $\rho : x + 3y + 2z - 10 = 0$.

$$[\, T_2 = \left[\frac{1}{4}, -\frac{1}{3}, \frac{1}{2}\right]; t_2 : x = \frac{1}{4} - r, y = -\frac{1}{3} + r, z = \frac{1}{2} - r, r \in \mathbb{R};$$

$$T_3 = \left[4, -\frac{8}{3}, 2\right], t_3 : x = 4 - 8s, y = -\frac{8}{3} + 4s, z = 2 - 2s, s \in \mathbb{R}\,]$$

$\gamma:\ x = x(t),\ y = y(t),\ z = z(t)$, $t \in \langle \alpha, \beta \rangle$ $\qquad T = [x_T, y_T, z_T]$

$t:\ x = x_T + x_t'(T)\cdot s,\ y = y_T + y_t'(T)\cdot s,\ z = z_T + z_t'(T)\cdot s$, $s \in \mathbb{R}$

$\mu:\ x_t'(T)(x - x_T) + y_t'(T)(y - y_T) + z_t'(T)(z - z_T) = 0$

$\vec{n}_\rho = (1, 2, 3)$

$\vec{s}_t = (x_t', y_t', z_t') = (\frac{4}{4}t^3, \frac{3}{3}t^2, \frac{2}{2}t) = (t^3, t^2, t)$

$\vec{n}_\rho \cdot \vec{s}_t = 0$ (JSOU KOLMÉ)

$t^3\cdot 1 + t^2\cdot 3 + t\cdot 2 = 0$

$t(t^2 + 3t + 2) = 0 \Rightarrow t_1 = 0$

$t_{2,3} = \frac{-3 \pm \sqrt{3^2 - 4\cdot 1\cdot 2}}{2\cdot 1} \left\langle \begin{matrix} -1 \\ 2 \end{matrix} \right.$

$T_1 = [0, 0, 0]$ PRO $t = 0$ $\rightarrow \ominus$

$T_2 = [\frac{1}{4}, -\frac{1}{3}, \frac{1}{2}]$ PRO $t = -1$

$T_3 = [4, -\frac{8}{3}, 2]$ PRO $t = -2$

$T_2:\ \vec{s}_2 = (-1, 1, -1)$ $\qquad T_3$: DÚ ...

$t_2:\ x = \frac{1}{4} - s,\ y = -\frac{1}{3} + s,\ z = \frac{1}{2} - s$, $s \in \mathbb{R}$